Sheet 4

Exercise 1 Let $n \geq 1$ be an integer.
Show any compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ is conjugated to a subgroup of $\mathrm{O}(n, \mathbb{R})$.
Hint : Construct a left invariant measure using invariant s-form on such a compact group.

Exercise 2 Recall that the Heisenberg group, Heis(3), consists of upper triangular matrices in $\mathcal{M}_3(\mathbb{R})$ with all diagonal elements equal to 1.
Denote by Γ the subgroup generated by $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Let $Z = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathfrak{heis}(3)$.

1. Let $\varphi : \mathfrak{heis}(3) \to \mathfrak{gl}_n(\mathbb{R})$ be a Lie algebra morphism. Show that $\varphi(Z)$ is nilpotent.
2. Deduce that $\{\exp(t\varphi(Z)) \mid t \in \mathbb{R}\}$ is either trivial or unbounded.

Let $\rho : \mathrm{Heis}(3)/\Gamma \to \mathrm{GL}_n(\mathbb{R})$ be a Lie group morphism.

3. Show that the one-parameter sugbroup $(\exp(td_e\rho(Z)))_t$ is compact.
4. Deduce that ρ is not injective.
   In other words, $\mathrm{Heis}(3)/\Gamma$ has no faithful linear representation.

Bracket between vector fields, and the exponential map

Exercise 3 Let X, Y be two vector fields on a manifold M and φ_t, ψ_t the local flows of X and Y.

1. Show that $[X, Y] = \frac{d}{dt}\Big|_{t=0} (\varphi_t)^*(Y)$.
2. Show that, for all $x \in M$, $\frac{d}{dt}\Big|_{t=0} \psi_t \circ \varphi_t \circ \psi_{-t} \circ \varphi_{-t}(x) = 0$.
3. Show that, for all $x \in M$, $[X, Y](x) = \frac{d}{dt}\Big|_{t=0^+} \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}}(x)$.

Let X, Y be two elements in the Lie algebra $\mathfrak{g}$ of a Lie group G, and $\exp : \mathfrak{g} \to G$ the exponential map of G.

4. Show that $\frac{d}{dt}\Big|_{t=0} \exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = 0$.
5. Show that $[X, Y] = \frac{d}{dt}\Big|_{t=0^+} \exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$.