Mathematics
Faculté des sciences d'Orsay

Sheet 3

Groups and geometry
M2 AAG 2025–2026

Universal cover of the group $\mathrm{SL}_2(\mathbb{R})$

Exercise 1 Denote by $\mathrm{SL}_2(\mathbb{R})$ the group of 2×2 matrices of determinant $+1$ with real entries.

1. Show that any one-parameter subgroup $(g^t)_{t\in\mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{R})$ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to one of the following three subgroups:
$$\left(s^t = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}\right)_{t\in\mathbb{R}}, \quad \left(u^t = \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix}\right)_{t\in\mathbb{R}}, \quad \left(r^t = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}\right)_{t\in\mathbb{R}}.$$
If a is nonzero, $(g^t)_{t\in\mathbb{R}}$ is called *hyperbolic* in the first case, *parabolic* in the second case, *elliptic* otherwise.
2. If $g^t = \exp(tZ)$ with $Z \in \mathfrak{sl}_2(\mathbb{R})$, relate the hyperbolic, parabolic or elliptic nature of (g^t) to the sign of $B(Z,Z)$, where B is the Killing form defined for every $X, Y \in \mathfrak{sl}_2(\mathbb{R})$ by
$$B(X,Y) = \frac{1}{2}\mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$
3. Is the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ surjective?

Denote by $\pi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathrm{SL}_2(\mathbb{R})$ the universal cover of $\mathrm{SL}_2(\mathbb{R})$, and by $\widetilde{\exp}$ the exponential map from $\mathfrak{sl}_2(\mathbb{R})$ to $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

4. What is the relation between $\exp$, $\widetilde{\exp}$ and π?
5. Describe the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
6. Show that if $B(Z,Z) < 0$, then $(\widetilde{\exp}(tZ))_{t\in\mathbb{R}}$ is an embedded Lie subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, isomorphic to $\mathbb{R}$ and containing the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
7. Are there any connected compact subgroups besides $\{e\}$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$?

Lie groups and Lie algebra correspondence, covers

Exercise 2 Classify up to isomorphism

1. all connected Lie groups with $\mathbb{R}^n$, where $n \geq 1$, as their Lie algebra;
2. all connected Lie groups with $\mathfrak{aff}(\mathbb{R})$ as their Lie algebra;
3. all connected Lie groups with $\mathfrak{heis}(3)$ as their Lie algebra.

Exercise 3 Let G be a Lie group with Lie algebra $\mathfrak{g}$.

1. Let H be a closed subgroup of G.
Show that if H is not discrete, it contains a nontrivial one-parameter subgroup.
2. Denote by $\mathfrak{z}(\mathfrak{g})$ and $Z(G)$ the centres of the Lie algebra $\mathfrak{g}$ and of the Lie group G.
Assuming G is connected, show that $Z(G)$ is discrete if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
3. Give an example of a connected Lie group G such that $\mathfrak{z}(\mathfrak{g}) = \{0\}$ but $Z(G) \neq \{e\}$.