Mathematics
Faculté des sciences d'Orsay

Sheet 2

Groups and geometry
M2 AAG 2025–2026

Left invariant vector fields and Lie derivatives

Exercise 1 Let $n \in \mathbb{N}$.

1. Prove that $\mathrm{GL}(n, \mathbb{R})$ is a Lie group, compute its Lie algebra.
2. Prove that any continuous map $X : \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})$ is a vector field for the manifold $\mathrm{GL}(n, \mathbb{R})$.
3. Prove that a left-invariant vector field is uniquely determined by its image at identity.
4. Let X and Y be left-invariant vector fields. Prove that $\mathcal{L}_X Y(e) = [X(e), Y(e)]$, where e is the identity element of $\mathrm{GL}(n, \mathbb{R})$. Is the vector field $\mathcal{L}_X Y$ also left-invariant?
   Hint: the flows φ_X^t and φ_Y^t are left-invariant isomorphisms of $\mathrm{GL}(n, \mathbb{R})$.

About the exponential map

Exercise 2 Let G be a real or complex Lie group and e its identity element.
Show there exists a neighbourhood U of e in G such that if H is a subgroup of G contained in U, then $H = \{e\}$.

Exercise 3 Let G and H be Lie groups. Assume G is connected.

1. Consider φ and ψ two Lie group morphisms from G to H. Assume there exists $g_0 \in G$ such that $\varphi(g_0) = \psi(g_0)$ et $T_{g_0}\varphi = T_{g_0}\psi$.
   Show that $T_e\varphi = T_e\psi$, then that φ and ψ coincide on a neighbourhood of g_0. Finally, conclude that $\varphi = \psi$.
2. Let φ be a C^∞ diffeomorphism of G, such that for any left-invariant vector field X, we have $\varphi^* X = X$. Show that there exists $g_0 \in G$ such that $\varphi = L_{g_0}$.

Some connected, some disconnected Lie groups

Exercise 4 Prove that if a subgroup H of a group G is connected and the quotient G/H is also connected, then G is connected.
Let $n \in \mathbb{N}$. Prove that the following groups are connected.

1. $\mathrm{GL}(n, \mathbb{R})^+ = \{g \in \mathcal{M}_n(\mathbb{R}) \mid \det g > 0\}$.
2. $\mathrm{SL}(n, \mathbb{R})$.
3. $\mathrm{SO}(n, \mathbb{R})$.

Exercise 5 Lorentz transformations
In $\mathbb{R}^{2,1}$ consider the quadratic form $Q\begin{pmatrix} x \\ y \\ t \end{pmatrix} = x^2 + y^2 - t^2$. Denote by (e_x, e_y, e_t) the canonical basis.

1. Draw the hypersurfaces $\{Q = 0\}$ and $\{Q = 1\}$ and $\{Q = -1\}$.

Vectors of $\mathbb{R}^{2,1}$ in $\{Q=0\}$ (resp. $\{Q>0\}$ and $\{Q<0\}$) are called light-like (resp. space-like and time-like).

2. The subsets of light-like (resp. space-like, time-like) vectors, which is connected?
3. Prove that the following map is surjective. Deduce that $\mathrm{SO}(2,1)$ is not connected.

$$\begin{aligned} \mathrm{SO}(2,1) &\longrightarrow \{Q=-1\} \\ g &\longmapsto ge_t. \end{aligned}$$

   Hint: one may want to use the following Q preserving transformations

$$\left\{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}, \quad \begin{pmatrix} 1 & & (0) \\ & -1 & \\ (0) & & -1 \end{pmatrix}.$$

Adjoint representation and invariant differential forms

Exercise 6 Let G be a Lie group of dimension $n \geq 1$, with Lie algebra $\mathfrak{g}$.

1. Show that the differential s-forms ω on G that are left-invariant (i.e. such that for any $g \in G$, $L_g^*(\omega) = \omega$), form a vector space in natural bijection with $(\Lambda^s(\mathfrak{g}))^*$.
2. Let ω be a left-invariant s-form on G. Show that for any $g \in G$, $R_g^*(\omega)$ is left-invariant. What element in $\Lambda^s(\mathfrak{g})$ does it correspond to?
3. Show that if G is compact and connected, then any left-invariant or right-invariant n-form is automatically bi-invariant.

On compact connected complex Lie groups

Exercise 7 Let G be a connected compact complex Lie group. Let V be a finite-dimensional complex vector space. Let $\rho : G \to \mathrm{GL}(V)$ be a complex Lie group morphism.

1. Show that $\rho(G) = \{\mathrm{Id}\}$.
   Hint: A bounded holomorphic map from $\mathbb{C}$ to $\mathbb{C}$ is constant.
2. Deduce that any connected compact complex Lie group is commutative.