Mathematics
Faculté des sciences d'Orsay

Sheet 1

Groups and geometry
M2 AAG 2025–2026

Exercise 1 Consider $\mathrm{Aff}(\mathbb{R}) = \{x \mapsto ax+b \ : \ (a,b) \in \mathbb{R}^* \times \mathbb{R}\}$ the set of affine transformations of $\mathbb{R}$.

1. Prove that $\mathrm{Aff}(\mathbb{R})$ is a Lie group.
2. Compute the Lie algebra of $\mathrm{Aff}(\mathbb{R})$, that we denote by $\mathfrak{aff}(\mathbb{R})$.
   Hint: One may write $\mathrm{Aff}(\mathbb{R})$ *as a subgroup of a matrix group.*

Let V be a real vector space of dimension 2, and let $\{X, Y\}$ be a basis of V.

3. Show that any anticommutative bilinear map on V defines a Lie bracket.
4. Show any 2-dimensional real Lie algebra is either commutative or isomorphic to $\mathfrak{aff}(\mathbb{R})$.
5. Check that each of these Lie algebras is isomorphic to the Lie algebra of a Lie group.

Exercise 2 Define the lower central series (C_n) and the derived series (D_n) of a Lie algebra $\mathfrak{g}$ by setting $C_0 = D_0 = \mathfrak{g}$ and, for any integer $n \geq 0$, $C_{n+1} = [C_n, \mathfrak{g}]$ and $D_{n+1} = [D_n, D_n]$.
Call $\mathfrak{g}$ nilpotent if there exists n such that $C_n = 0$; solvable if there exists n such that $D_n = 0$.
For each group below:

(a) show it is a Lie group;
(b) tell if it is connected or not;
(c) compute if possible its Lie algebra $\mathfrak{g}$;
(d) give a basis B for its Lie algebra $\mathfrak{g}$;
(e) compute brackets between elements in B;
(f) is $\mathfrak{g}$ commutative? nilpotent? solvable?

1. $\mathrm{O}_2(\mathbb{R}) = \{x \in \mathcal{M}_2(\mathbb{R}) \,|\, {}^t x x = I_2.\}$
2. $\mathrm{SL}_2(\mathbb{R}) = \{x \in \mathcal{M}_2(\mathbb{R}) \,|\, \det x = 1\}$
3. $\mathrm{SU}(2) = \{x \in \mathcal{M}_2(\mathbb{C}) \,|\, {}^t\overline{x}x = I_2 \ \& \ \det x = 1\}$
4. $\mathrm{Heis}(3) = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| (x,y,z) \in \mathbb{R}^3 \right\}.$
5. $\mathrm{Sol}(3) = \mathbb{R} \ltimes \mathbb{R}^2$, with the product manifold structure, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. (Realise $\mathrm{Sol}(3)$ as an embedded subgroup in $\mathrm{SL}_3(\mathbb{R})$.)

Exercise 3 *Isomorphisms* $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ *et* $\mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{so}(1,2)$.
Recall the following definitions of classical Lie groups

$$\mathrm{SO}(3) = \{x \in \mathcal{M}_3(\mathbb{R}) \ : \ {}^t x\, x = I_3 \ \& \ \det x = 1\}$$
$$\mathrm{SO}(2,1) = \left\{ x \in \mathcal{M}_3(\mathbb{R}) \ : \ {}^t x \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} x = \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} \ \& \ \det x = 1 \right\}$$

1. Check that $\mathrm{SU}(2)$ consists in matrices of the form $\left(\begin{array}{cc} a & b \\ -\overline{b} & \overline{a} \end{array} \right)$, with $|a|^2 + |b|^2 = 1$.
   Deduce that $\mathrm{SU}(2)$ is diffeomorphic to a well-known 3-manifold.

2. Show that $E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
   form a basis of its Lie algebra $\mathfrak{su}(2)$.
   What are the bracket relations between E_1, E_2 et E_3.

On $\mathfrak{su}(2)$, consider the quadratic form $Q(M) = \det(M)$.

3. Let $\varphi : \mathfrak{su}(2) \to \mathbb{R}^3$ give the coordinates of $U \in \mathfrak{su}(2)$ in the basis $\{2E_1, 2E_2, 2E_3\}$.
   What quadratic form on $\mathbb{R}^3$ corresponds to Q via φ?
4. Deduce that the adjoint representation naturally defines
   a morphism $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$.
5. Show that ρ is a submersion, then that it is a surjective morphism.
   What is its kernel?
6. Deduce from the above that $\mathrm{SO}(3)$ is diffeomorphic to a well-known 3-manifold
   and that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

7. Show that H, E and F forme a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.
   Compute the brackets between H, E, F.

Given a finite-dimensional real Lie algebra $\mathfrak{g}$, its Killing form
is the symmetric bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $B_{\mathfrak{g}}(x, y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$.

8. Show that a Lie algebra isomorphism sends Killing form to Killing form.
9. Denote by B the Killing form on $\mathfrak{sl}_2(\mathbb{R})$.
   Find the matrix of B in the basis H, E, F.
   What is its signature?
10. Determine the signature of the Killing form of $\mathfrak{su}(2)$.
    Deduce that the Lie algebras $\mathfrak{su}(2)$ et $\mathfrak{sl}_2(\mathbb{R})$ are not isomorphic.
11. Adapt the above to show the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ et $\mathfrak{so}(1, 2)$ are isomorphic.